SUGGESTED SOLUTIONS TO HOMEWORK 2

Exercise 1 (6.2.14)**.** Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if the derivative f' is never 0 on I, then either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

Proof. Since f' is never 0 on I, let us further assume there exist $a < b \in I$ such that $f'(a) < 0 < f'(b)$, then by Darboux's Theorem, there exists $c \in (a, b)$ such that $f'(c) = 0$, which is a contradiction. Therefore either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$. □

Exercise 2 (6.2.16)**.** Let $f : (0, \infty] \to \mathbb{R}$ be differentiable on $(0, \infty)$ and assume that $f'(x) \to b$ as $x \to \infty$.

(a) Show that for any $h > 0$, we have $\lim\limits_{x\to\infty} \frac{f(x+h)-f(x)}{h} = b$.

(b) Show that if $f(x) \to a$ as $x \to \infty$, then $b = 0$.

(c) Show that $\lim\limits_{x\to\infty} \frac{f(x)}{x} = b$.

Proof. (a) Given $\epsilon > 0$, there exists $A > 0$ such that for $x > A$, we have

$$|f'(x) - b| < \epsilon.$$

Therefore for any $h > 0$, let $x > A$, then by Mean Value Theorem, there exists $x_h \in (x, x+h)$ such that

$$|\frac{f(x+h) - f(x)}{h} - b| = |f'(x_h) - b| < \epsilon,$$

(b) Assume that $b \neq 0$ and let $\epsilon < \frac{|b|}{2}$, then there exists $A_1 > 0$ such that for $x > A_1$, we have

$$|f'(x) - b| < \frac{|b|}{2},$$

which implies

$$|f'(x)| > \frac{|b|}{2},$$

for $x > A_1$. Since $\lim\limits_{x\to\infty} f(x) = a$, then there exist $A_2 > 0$ such that for $y > x > A_2$, we have

$$|f(x) - f(y)| < \epsilon.$$

Let $x > \max\{A_1, A_2\}$ and $y = x + 1$, By Mean Value Theorem, there exists $x_0 \in (x, x + 1)$ such that

$$|f'(x_0)| < \epsilon,$$

which is a contradiction. Therefore $b = 0$.

(c) Given $\epsilon > 0$, let $A' > A$ and $x > \max\{A, \frac{f(A')}{\epsilon}, \frac{f'(x_0)A'}{\epsilon}\}$, then by Mean Value Theorem, there exists $x_0 \in (A', x)$ such that

$$f(x) - f(A') = f'(x_0)(x - A'),$$

therefore

$$|\frac{f(x)}{x} - b| \leq |\frac{f(A')}{x}| + |f'(x_0) - b| + |\frac{f'(x_0)A'}{x}| < 3\epsilon,$$

which implies $\lim_{x\to\infty} \frac{f(x)}{x} = b$. $\square$

Exercise 3 (6.3.1)**.** Suppose that f and g are continuous on $[a,b]$, differentiable on (a,b), that $c \in [a,b]$ and $g(x) \neq 0$ for $x \in [a,b]$, $x \neq c$. Let $A := \lim_{x\to c} f$ and $B := \lim_{x\to c} g$, and if $B = 0$, and $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists in $\mathbb{R}$, show that we must have $A = 0$.

Proof. Interchanging the order of limit and product,

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) \cdot \lim_{x\to c} \frac{f(x)}{g(x)} = 0,$$

which implies $A = 0$. $\square$

Exercise 4 (6.3.2)**.** In addition to the preceding exercise, let $g(x) > 0$ for $x \in [a,b]$, $x \neq c$. If $A > 0$ and $B = 0$, prove that we must have $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$. If $A < 0$ and $B = 0$, prove that we must have $\lim_{x\to c} \frac{f(x)}{g(x)} = -\infty$.

Proof. It suffices to prove the first case. For arbitrary $C > 0$, let $\epsilon < \frac{A}{C+1}$, then there exists $\delta > 0$ such that for $x \in (c-\delta, c+\delta)$, we have

$$|f(x) - A| < \epsilon, \quad |g(x)| < \epsilon,$$

therefore

$$|\frac{f(x)}{g(x)}| > \frac{A-\epsilon}{\epsilon} > C,$$

which implies that $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$. $\square$

Exercise 5 (6.3.4)**.** Let $f(x) := x^2$ for x rational, let $f(x) := 0$ for x irrational, and let $g(x) := \sin x$ for $x \in \mathbb{R}$. Use Theorem 6.3.1 to show that $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$. Explain why Theorem 6.3.3 cannot be used.

Proof. Let us first show that f' exists at $x = 0$. For arbitrary $\epsilon > 0$, let $A < \epsilon$, for $|h| < A$, if h is rational, we have

$$|\frac{f(h)}{h}| < |h| < \epsilon,$$

if h is irrational, we have

$$|\frac{f(h)}{h}| = 0.$$

Therefore $f'(0) = 0$. By Theorem 6.3.1, we have $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.

It should be noted that Theorem 6.3.3 cannot be used. Because f' does not exists for $x \neq 0$. Indeed, f is not continuous except $x = 0$. $\square$

Exercise 6 (6.3.13)**.** Try to use L'Hospital's Rule to find the limit of $\frac{\tan x}{\sec x}$ as $x \to (\frac{\pi}{2})-$. Then evaluate directly by changing to sines and cosines.

Proof. On the one hand, by L'Hospital's Rule,

$$\lim_{x\to(\frac{\pi}{2})-} \frac{\tan x}{\sec x} = \lim_{x\to(\frac{\pi}{2})-} \frac{\sec^2 x}{\sec x \tan x} = \lim_{x\to(\frac{\pi}{2})-} \frac{1}{\sin x} = 1.$$

On the other hand,

$$\lim_{x \to (\frac{\pi}{2})-} \frac{\tan x}{\sec x} = \lim_{x \to (\frac{\pi}{2})-} \sin x = 1.$$

$\square$